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Multiple Steady States, Poverty Traps and
Indeterminacy in the Uzawa–Lucas Model
with Small and Large Educational Externality

Shiro Kuwahara

School of Economics
University of Hyogo

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Institute for Policy Analysis and Social Innovation
University of Hyogo
Nishiku, Kobe 651-2197, Japan

Multiple Steady States, Poverty Traps and Indeterminacy in the Uzawa–Lucas Model with Small and Large Educational Externality*

Shiro Kuwahara[‡]
University of Hyogo

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Abstract

This study attempts to comprehensively explain economic growth and stagnation in advanced and developing countries by endogenizing educational efficiency, the critical exogenous parameter in the Uzawa–Lucas model. In particular, we examine the dynamic properties of educational efficiency given its substantial role in yielding a long-run growth rate. The model yields multiple steady states under an intertemporal substitution elasticity that is greater than 1. The results reveal that a steady state with a higher growth rate shows indeterminacy, and the identification of the steady states depends on how expectations are formulated. Thus, realizing long-run growth with a higher growth rate can be difficult owing to expectation formulations.

Keywords: Uzawa–Lucas model; Educational Externality; Multiple Steady States; Indeterminacy.

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[†]E-mail address: kuwahara@econ.u-hyogo.ac.jp

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1 Introduction

This study attempts to develop a simple extension of the Uzawa-Lucas model (Uzawa 1965, Lucas 1988) and to provide an explanation for positive growth, no growth, and the multiplicity of economic paths. Economic growth theory focuses on realizing long-term growth and considers human capital as one of the key inputs for such growth. Given its simplicity and convenience, the Uzawa-Lucas model has been analyzed by introducing certain externalities; for example, Mulligan and Sala-i-Martin (1993), Benhabib and Rerli (1994), Xie (1994), and Gomez (2003, 2004). Although these externalities are typically introduced in the goods production sector, this study introduces the externality in the human capital accumulation sector. This setup yields not only an endogenously determined long-run growth factor but also a multiplicity of economic paths, which would explain the diversity of economic paths that are described, for example, as the "mystery of economic growth" in Lucas (1988), which is a seminal paper on endogenous growth theory.

Furthermore, the present analysis of the multiplicity of economic paths also sheds light on stagnations caused by large economic shocks and the economic booms that precede the shocks. Of course, these phenomena are understood as monetary ones, (see, for example, Kindeleberger (1978)); however, in the preceding expansionary term, we can observe TFP (total factor productivity) growth (see, for example, Kunieda and Shibata (2012)). The present study aims to replicate multiple economic paths, such as one with a high growth-rate path with the property of local indeterminacy and one with a low- or no-growth rate path with saddle point stability.

In the Uzawa–Lucas model, the exogenously given efficiency parameter of human capital accumulation (i.e., the educational efficiency parameter), and the linearity of human capital investment are critical determinants of the long-run growth rate. By denoting the educational efficiency parameter by b , the subjective discount parameter by ρ , and the intertemporal substitution elasticity by $1/\theta$, we obtain the long-run growth rate as $(1/\theta)(b-\rho)$. Furthermore, if $b > \rho$, then this long-run growth is realized, and if $b < \rho$, no human capital accumulation is experienced and the economy is stuck in a no-growth trap. Thus, the realization of long-run growth is contingent on exogenously given parameter restrictions in the normal Uzawa–Lucas model.¹

By contrast, the present model assumes that the educational efficiency parameter b is endogenously determined but the dynamics are exogenous for economic agents. Therefore, we call this inserted mechanism, "educational

¹In this case, Kuwahara (2013) is an exception, where an exogenous b is assumed, but long-run growth both with and without human capital investment is generated from an international knowledge spillover.

externality,” and extend the Uzawa–Lucas framework by introducing the dynamics of educational externality efficiency. For simplification, we consider educational efficiency as a function of the economic production level captured by physical capital endowment per human capital accumulation. As is broadly recognized, education is an engine for economic development, which implies that economic accumulation is generally accompanied by the educational system. Thus, it is natural for a household to regard the educational environment as given rather than a private decision. Therefore, we can say that we develop a Uzawa–Lucas model with the minimum and simplest relationship between educational efficiency level and economic growth to present the interrelationship between the two.

The results are summarized as follows. Too large educational externality makes economy explosive. The model contains multiple steady states with intertemporal substitution elasticity over 1², and there is a no-growth steady state in the case of lower educational efficiency and a higher subjective discount rate. The selection between the two depends on expectation formation, although the steady state with a high growth rate shows indeterminacy properties under a small externality. Thus, it can be considered that the former corresponds to advanced economies because of more efficient education and long-run positive growth, while the latter to developing economies given the less efficient education level and absence of a growth steady state. Therefore, the difficulties of maintaining high growth for advanced economies and achieving positive growth or initiating the growth path for developing economies are caused by expectation formation under indeterminacy.

This paper is organized as follows. Section 2 presents the model. Section 3 derives the steady states. Section 4 discusses the dynamics and stability of the system. Section 5 provides concluding remarks.

2 Model

2.1 Model with externality on educational efficiency

We assume a normal Uzawa–Lucas-type final goods production structure. It is constructed using physical capital (per capita capital stock is denoted by k) and human capital (per capita human capital stock is h) and is consumed as consumption goods (per capita consumption is c) and invested by physical capital (increment of per capita capital stock is \dot{k}); the final goods market

²Note that intertemporal substitution elasticity larger than 1 is necessary to generate multiplicity in the present study to generate multiplicity and this condition is supported by some empirical studies, for example, Vissing-Jorgenson and Attanasio (2003).

is competitive. The division rate of human capital in goods production is denoted as $u(\in (0, 1])$; therefore, that of human capital in education is $1 - u$.

The clearing condition of the final goods market provides the dynamic equation of k as follows:

$$\dot{k}(t) = \underbrace{Ak(t)^\alpha(u(t)h(t))^{1-\alpha}}_{y(t)} - c(t), \quad 0 < \alpha < 1, \quad (1)$$

where $A(> 0)$ is the efficiency parameter of final goods production.

Human capital is assumed to be accumulated through the following equation:

$$\dot{h}(t) = b(t)(1 - u(t))h(t), \quad b > 0, \quad (2)$$

where $b(t)$ is an efficiency parameter of human capital assumed to be a variable and the dynamics are introduced later in the paper. Furthermore, we assume that the household regards the dynamics as exogenously determined.

The representative household is assumed to have the following dynamical objective function on utility maximization:

$$\max \int_0^\infty \frac{c(t)^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt, \quad (3)$$

where c , $\theta(> 0)$, and $\rho > 0$ are per capita consumption, the constant relative risk aversion (CRRA) parameter, and subjective discount rate, respectively. Note that the CRRA parameter corresponds to the reciprocal of intertemporal substitution elasticity in this class of utility functions.

Considering the dynamics of $b(t)$ as exogenous, the household maximizes this subject to the budget constraint, and the optimal conditions in the case of positive human capital investment, we call the "Uzawa regime," are calculated as follows:

$$\lambda(t) = c(t)^{-\theta}, \quad (4)$$

$$\lambda(t)(1 - \alpha) \frac{y(t)}{u(t)} = \mu(t)b(t)h(t), \quad (5)$$

$$\rho\lambda(t) - \dot{\lambda}(t) = \frac{\partial \mathcal{H}}{\partial k(t)} = \lambda(t)\alpha \frac{y(t)}{k(t)}, \quad (6)$$

$$\rho\mu(t) - \dot{\mu}(t) = \frac{\partial \mathcal{H}}{\partial h(t)} = \lambda(t)(1 - \alpha) \frac{y(t)}{h(t)} + \mu(t)b(t)(1 - u(t)), \quad (7)$$

$$\lim_{t \rightarrow \infty} e^{-\rho t} \lambda(t)k(t) = 0, \quad \text{and} \quad \lim_{t \rightarrow \infty} e^{-\rho t} \mu(t)h(t) = 0, \quad (8)$$

where λ and μ are the shadow prices of physical and human capital.

Then, we derive the dynamical equations in the Uzawa regime. Using (4), (5), (6), and (7), we derive the following equations:

$$\rho - \frac{\dot{\lambda}(t)}{\lambda(t)} = \alpha \frac{y(t)}{k(t)} \quad (:= r(t)), \quad (9)$$

$$\rho - \frac{\dot{\mu}(t)}{\mu(t)} = \frac{\lambda(t)}{\mu(t)}(1 - \alpha) \frac{y(t)}{h(t)} + b(t)(1 - u(t)) = b(t), \quad (10)$$

Eqs.(6) and (7), and therefore, Eqs.(9) and (10), respectively, denote the optimal conditions for physical and human capital. r and b represent the marginal rate of transformation of physical and human capital, and as shown later, the growth rates of λ and μ and values of r and b are equated in the steady states with positive human capital accumulation.

Combining (4) and (9) yields the following Euler equation:

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta} \{ \alpha A x(t)^{\alpha-1} u(t)^{1-\alpha} - \rho \} = \frac{1}{\theta} (r(t) - \rho), \quad (11)$$

where $x := k/h$. Using (5), (9), and (10), we get

$$\frac{\dot{u}(t)}{u(t)} = \frac{1}{\alpha} \left[b(t) - r(t) + \alpha \frac{\dot{x}(t)}{x(t)} - \frac{\dot{b}(t)}{b(t)} \right]. \quad (12)$$

Here, we introduce an assumption regarding the dynamics of b . For educational externality, we assume that the human capital endowment has a positive spillover on educational productivity, and thus, $\frac{\partial b}{\partial h} > 0$. Next, we need the property in which the effects are stationary in the steady states, and thus, additionally assume that the larger the economy, the smaller the spillover. We also assume the scale of an economy is captured by the per capita capital stock k , and thus, $\frac{\partial b}{\partial k} < 0$. Therefore, b is assumed as follows:

Assumption The dynamics of b negatively depends on $x = k/h$:

$$b(t) = b(x(t)), \quad b'(\cdot) < 0.$$

Furthermore, we assume that the elasticity of b on x , denoted by $\varepsilon (> 0)$, is constant: $\varepsilon := -\frac{b'(x)x}{b(x)} = (const)$. Thus, we specify the form of $b(x)$ as follows: $b(x) = \bar{b}x(t)^{-\varepsilon}$, where $\bar{b} (> 0)$ is the total efficiency of human capital accumulation and elasticity ε captures the externality intensity of k and h . A larger ε means larger externality of human capital accumulation. Note that the assumption $\varepsilon = 0$ makes the present model the normal Uzawa–Lucas model.

3 Steady states

First, we consider the steady state with positive human capital accumulation, or the inner solution case, $u^* \in (0, 1)$. We call this case the ‘‘Uzawa regime.’’ A balanced growth path in the Uzawa regime is the state in which k , h , and c grow at a constant rate. In addition, u is constant and $u \in (0, 1)$, and therefore, x , q , and u are constant. We analyze the properties of steady states using three variable sets $\{x^*, q^*, u^*\}$, where the index $*$ denotes the value at the steady state.

(1) and (2) imply that $g_y^* = g_k^* = g_c^* = g_h^* = b(x^*)(1 - u^*)$ ($:= g^*$) and (4) and (5) imply $-\theta g_c^* = g_\mu^* = g_\lambda^*$. Therefore, by substituting $g^* = -(1/\theta)g_\mu^*$ into (10), we get

$$(1 - u^*)b(x^*)\theta = b(x^*) - \rho. \quad (13)$$

This relationship provides two equations: one is an equilibrium relationship between x and u and the other is the equilibrium growth rate. The former can be immediately transformed from (13) and is given as

$$u^* = 1 - \frac{1}{\theta} + \frac{\rho}{\theta b(x^*)} (:= \Psi(x^*)). \quad (14)$$

Imposing the condition in a steady state ($\dot{x} = 0$ and $\dot{u} = 0$) on (12), we have

$$b(x^*) = \alpha A x^{*\alpha-1} u^{*1-\alpha}. \quad (15)$$

Solving (15) with respect to u^* , we obtain the following equation that shows the relationship between x and u in a steady state:

$$u^* = \left(\frac{b(x^*)}{\alpha A} \right)^{\frac{1}{1-\alpha}} x^* (:= \Phi(x^*)). \quad (16)$$

The intersection of the two equations, Ψ and Φ , determines the equilibrium value(s) of u^* and x^* . Under the specification of $b(x) = \bar{b}x^{-\varepsilon}$, we have

$$u = \Psi(x) = \frac{\theta - 1}{\theta} + \underbrace{\frac{\rho}{\theta \bar{b}}}_{:= \bar{\Psi}(x)} x^\varepsilon, \quad u = \Phi(x) = \left[\frac{\bar{b}}{\alpha A} \right]^{\frac{1}{1-\alpha}} x^{1-\frac{\varepsilon}{1-\alpha}},$$

where $\bar{\Psi}$ is the variable term of Ψ . These two functions have the following properties: Ψ is an increasing function and $\Phi(0) = 0$, but the gradient of Φ depends on the externality level ε , and the sign of $\Psi(0)$ depends on the reciprocal of the elasticity of the intertemporal substitution parameter θ .

Because the intersection of Φ and Ψ is so diverse, it is enormous and diffusive to observe all cases. Therefore, we make certain assumptions to confine the cases to an appropriate degree for the purpose of this study. To understand the shift from ε to Φ and Ψ , Fig. 1 is an ad referendum drawn under the assumption $\theta < 1$, and $1 - \frac{1}{\theta} + \frac{\bar{b}}{\theta\rho} > 0$.³ Note that if $\varepsilon \rightarrow 0$, the model is reduced to the normal Uzawa–Lucas model, and $\frac{\theta-1}{\theta} + \frac{\bar{b}}{\theta\rho} > 0$ is one of the necessary conditions for the inner solution.

Next, we check the steady state with *no* human capital investment; therefore, there is no growth in the long run. We call this case the "Solow regime," where all human capital is employed in the final goods production sector; therefore, $u(t)$ becomes constant $u = 1$. In this case, human capital investment is not optimal for a household, and the model resembles the Solow model. Substituting $u = 1$ into the Euler equation, we obtain $r^{**} = \alpha Ax^{**\alpha-1} = \rho$, and therefore, $x^{**} = \left[\frac{\alpha A}{\rho}\right]^{\frac{1}{1-\alpha}}$, where $**$ denotes the steady state value of the Solow regime. The difference between the Solow model and Solow regime in this study is the existence of the non-profitable condition for human capital investment. This phenomenon emerges when the MRT of human capital is lower than that of physical capital; therefore, we have the following condition: $b(x^{**}) < \alpha Ax^{**\alpha-1} (= \rho)$, that is, $\bar{b} < \rho^{1-\frac{\varepsilon}{1-\alpha}}(\alpha A)^{\frac{\varepsilon}{1-\alpha}} (= \tilde{\rho})$. From this, we obtain the following Lemma:

Lemma 1-1

$$\bar{b} \left\{ \begin{array}{l} > \\ < \end{array} \right\} \tilde{\rho} \Leftrightarrow \text{Economy} \left\{ \begin{array}{l} \text{does not have} \\ \text{has} \end{array} \right\}$$

the Steady State in the Solow Regime.

We might term a country with $\bar{b} > \tilde{\rho}$ as "advanced country" and one with $\bar{b} < \tilde{\rho}$ as "developing country." The domain that the Solow regime steady state exists is depicted on \bar{b} - ρ plain in Fig. 2. Similar to the normal Uzawa–Lucas model, the educational efficiency parameter \bar{b} and subjective discount rate ρ determine the no-growth steady state. Under a more intense externality, the upward-sloping relationship between the parameters is disturbed and a small subjective discount and high educational efficiency yield poverty traps. In addition, the production parameter also affects the condition of no-growth traps. Higher parameters of production A and physical capital α (higher α decreases human capital efficiency in production $1 - \alpha$) make higher educational efficiency necessary for the existence of long-run positive growth; these affects relatively disadvantage education.

³In the interval of $\varepsilon \in (0, 1 - \alpha)$, Φ is always increasing.

To clear the relationship between the above condition and equations Ψ and Φ , we define $x = \Phi^{-1}(u)$ and $x = \Psi^{-1}(u)$ as the inverse function of $u = \Psi(x)$ and $u = \Phi(x)$ and $x_\phi := \Phi^{-1}(1) = \{\bar{b}/(\alpha A)\}^{-\frac{1}{1+\alpha+\varepsilon}}$ and $x_\psi := \Psi^{-1}(1) = (\bar{b}/\rho)^{1/\varepsilon}$. Then, we derive the following:

$$\left\{ \begin{array}{l} x_\phi < x^{**} < x_\psi \\ x_\psi < (x^{**}) < x_\phi \end{array} \right\} \iff \bar{b} \left\{ \begin{array}{l} < \\ > \end{array} \right\} \tilde{\rho}.$$

Next, we explore the steady state in the Uzawa regime. Here, we consider two cases: one with the Solow regime and the other without.

Case I: Small externality case ($\varepsilon \in (0, 1 - \alpha)$) At first, we inquire the case with small externality, specified as follows:⁴

$$0 < \varepsilon < \bar{\varepsilon} := \frac{1 - \alpha}{2 - \alpha}, \quad \left(\bar{\varepsilon} \in \left(0, \frac{1}{2} \right) \right).$$

. The phase of steady states with $\bar{b} > \tilde{\rho}$ and $\varepsilon \in (0, 1 - \alpha)$ is depicted on ρ - \bar{b} plane in Fig.3 (a).

To obtain the condition of steady states, we reconsider the relation between Φ and Ψ from the viewpoint of u , thus we obtain the equation $u = \Psi(\Phi^{-1}(u))$ which gives the equilibrium, and this equation is transformed into

$$(L(u) :=) u + \frac{1 - \theta}{\theta} = \frac{1}{\theta} \Omega^{\frac{1}{\beta+\varepsilon}} u^{\frac{\varepsilon}{\beta+\varepsilon}} (:= R(u)),$$

where $\beta := 1 - \frac{\varepsilon}{\varepsilon}$ and $\Omega := \frac{\tilde{\rho}}{\bar{b}}$. In the case discussed here, $\beta > 0$, $\Omega \in (0, 1)$ and $\frac{\varepsilon}{\beta+\varepsilon} \in (0, 1)$ respectively hold. We respectively define the LHS and RHS of the above equation as $L(u)$ and $R(u)$. The graph of $L(u)$ and $R(u)$ are given by Fig.4(a).

Lemma 1-2 *Under the assumption $\theta \in (\underline{\theta}, 1)$, $\bar{b} > \tilde{\rho}$, and $\varepsilon \in (0, \bar{\varepsilon})$, namely high intertemporal elasticity of substitution, high educational efficiency, and small educational externality, we always have multiple equilibria $\{E_1, E_2\}$ where both equilibria are related with long-run positive growth.*

⁴The properties of the intersection of Ψ and Φ are changed by ε , the intense parameter of externality, and the properties are divided by some thresholds such as $\bar{\varepsilon}$ and $1 - \alpha (> \bar{\varepsilon})$. For example, $1 - \alpha$ is the threshold that the incline of Φ is change from increasing to decreasing for the increment of ε . Then, the above assumption given in (R_1) implies that we focus on the case with small externality. To understand the implication of $\bar{\varepsilon}$, we check the gradient of these two lines. We can derive ε from the following equation: $\bar{\varepsilon} = \arg \left\{ \varepsilon \left[\left(\frac{\Psi'(x)x}{\Psi(x)} = \right) \varepsilon = 1 - \frac{\varepsilon}{1-\alpha} \left(= \frac{\Phi'(x)x}{\Phi(x)} \right) \right] \right\}$. Therefore, the $\bar{\varepsilon}$ is the point where Φ and Ψ has the same buckling, and $\varepsilon > (<) \bar{\varepsilon}$ implies that for small (large) externality, buckling of Φ is larger (smaller) than that of Ψ .

Proof) Since $L(0) > R(0)$ and $L(1) > R(1)$, which are respectively provided by $\theta < 1$ and $\bar{b} > \tilde{\rho}$, the case, in which L and R have two equilibrium u_1^* and u_2^* , is conditioned by $L(u) > R(u)$ for $\exists u \in (0, 1)$.

To show this, we define $\hat{u} := \arg\{u | R'(u) = 1 (= L'(u))\}$, namely \hat{u} is the point where the gradient of R is equaled to unity which is the gradient of L , and $L(\hat{u}) > R(\hat{u})$ is to be proved. Since $R'(\hat{u}) = 1$ and $R'(u) = \frac{\varepsilon}{\beta + \varepsilon} \frac{R(u)}{u}$, we have $R(\hat{u}) = \frac{\beta + \varepsilon}{\varepsilon} \hat{u} (= Q(\hat{u}))$. $R(\hat{u}) = Q(\hat{u})$ gives

$$\hat{u} = \left(\frac{\varepsilon}{\beta + \varepsilon} \right)^{\frac{\beta + \varepsilon}{\beta}} \left(\frac{1}{\theta} \right)^{\frac{\beta + \varepsilon}{\beta}} \Omega^{\frac{1}{\beta}}$$

Substituting the \hat{u} derived here into the condition $L(\hat{u}) > R(\hat{u})$, we obtain the condition $\hat{u} > \underline{u}$, where $\underline{u} := \frac{\varepsilon}{\beta} \frac{1 - \theta}{\theta}$. \hat{u} is the intersection of $Q(u)$ and $L(u)$. From the Fig.5(a), where \hat{u} and \underline{u} are drawn on $(1/\theta)$ - u plane, we have $\hat{u} > \underline{u}$ for $(1/\theta) > 1$, namely, $\theta < 1$, thus, $R(\hat{u}) > L(\hat{u})$. Thus, we have $u_2^* < \underline{u} < \hat{u} < u_1^*$.

From the necessary condition $\hat{u} < 1$, we have $\theta > \underline{\theta}$, where $\underline{\theta} := \frac{\varepsilon}{\beta + \varepsilon} \Omega^{\frac{1}{\beta + \varepsilon}}$ ($\in (0, 1)$). (Q.E.D)

It should be noted that $u_2^* < \underline{u} < \hat{u} < u_1^*$ also plays an important role in the determination of stability, which is discussed in the Ch.4 in this paper.

Here, we change the condition $\bar{b} > \tilde{\rho}$ in the above lemma into $\bar{b} < \tilde{\rho}$, then the equilibrium E_1 is vanished and E_0 alternatively emerges⁵. The phase of steady states are given in the Tab.1(b).

Case II: Middle externality Case ($\varepsilon \in (\bar{\varepsilon}, 1 - \alpha)$) Next, we inquire the case with middle externality. When we additionally adopt the assumptions; $\theta > 1$, and $\bar{b} < \tilde{\rho}$, we obtain the phase of steady states are depicted on ρ - \bar{b} plane in Fig.4 (b). In this case, we obtain the following lemma:

Lemma 1-3 *Under the assumption $\theta > 1$, $\bar{b} < \tilde{\rho}$, and $\varepsilon \in (\bar{\varepsilon}, 1 - \alpha)$, namely low intertemporal elasticity of substitution, low educational efficiency, and relatively large educational externality, we obtain multiple equilibria $\{E_0, E_1, E_2\}$ where both equilibria $\{E_1, E_2\}$ are related with long-run positive growth, and the equilibrium $\{E_0\}$ is related with no growth traps.*

⁵It should be noted that E_0 is always existing under $\bar{b} < \tilde{\rho}$.

Table 1: Equilibrium Set

(A) Case of $\bar{b} > \tilde{\rho}$			
θ	(a) $0 < \varepsilon < \bar{\varepsilon}$	(b) $\bar{\varepsilon} < \varepsilon < 1 - \alpha$	(c) $\varepsilon > 1 - \alpha$
$\theta < 1$	$\{E_1, E_2\}$	no steady state	$\{E_3\}$
$\theta > 1$	$\{E_1\}$	$\{E_2\}$	$\{E_3\}$

(B) Case of $\bar{b} < \tilde{\rho}$			
θ	(d) $0 < \varepsilon < \bar{\varepsilon}$	(e) $\bar{\varepsilon} < \varepsilon < 1 - \alpha$	(f) $\varepsilon > 1 - \alpha$
$\theta < 1$	$\{E_0, E_2\}$	$\{E_0, E_1\}$	$\{E_0\}$
$\theta > 1$	$\{E_0\}$	$\{E_0, E_1, E_2\}$	$\{E_0\}$

Proof) In this case, we obtain $\beta < 1$, $\Omega < 1$, and $\frac{\varepsilon}{\beta+\varepsilon} > 1$. Since $L(0) > R(0)$ and $L(1) > R(1)$ hold in this case, the property that L and R have two equilibrium u_1^* and u_2^* is conditioned by $L(u) > R(u)$ for $\exists u \in (0, 1)$.

In Case II, we also obtain the same threshold value \hat{u} and \bar{u} , depicted in Fig. 5(b), where $\left(\frac{\varepsilon}{\beta+\varepsilon}\right)^{\frac{\beta+\varepsilon}{\beta}} \Omega^{\frac{1}{\beta}} > 1$ is derived from $\frac{\varepsilon}{\beta+\varepsilon} > 1$ and $\Omega^{\frac{1}{\beta}} > 1$. Thus, we also obtain $u_2^* < \underline{u} < \hat{u} < u_1^*$. (Q.E.D)

Here, we also change the condition $\bar{b} < \tilde{\rho}$ in the above lemma into $\bar{b} > \tilde{\rho}$, then the equilibria E_0 and E_1 are vanished. The phase of steady states are given in the Tab.1(a).

Case III: Large externality case ($\varepsilon > 1 - \alpha$) The last case is the one with large externality. In this case, the function Φ becomes decreasing, so multiple steady states does not emerge. The pattern of steady states is determined by only the condition $\bar{b} > \tilde{\rho}$ or $\bar{b} < \tilde{\rho}$. The obtained result is given in Tab.1 in the column of $\varepsilon > 1 - \alpha$, and depicted in Fig. 3 (c) and (d). E_3 denotes the equilibrium with positive human capital accumulation in the Case III.

Finally, we check the TVC. From (8), we have $\rho \geq \gamma_\lambda^* + \gamma_k^*$ and $\rho \geq \gamma_\mu^* + \gamma_h^*$, where $\gamma_Z := \frac{\dot{Z}}{Z}$. Since $p = \mu/\lambda$ and $x = k/h$ are constant in steady states, both conditions are satisfied if one condition is shown to hold. Substituting $\gamma_\lambda = -\theta\gamma_c = -\theta g^* = -(b^* - \rho)$ and $g_k^* = g^* = b^*(1 - u^*)$ into $\rho \geq \gamma_\lambda^* + \gamma_k^*$ yields $b^*u^* \geq 0$, therefore, the steady states in the Uzawa regime obtained above always satisfy the TVC conditions.

Note that in the no-externality case, $\varepsilon = 0$, the condition becomes the same as that in the normal Uzawa–Lucas model. In the following section, we analyze the stability of the obtained steady states.

One of results obtained here is that an developing country might have more possibility for multiple steady states, which corresponds to the large fluctuation of growth dynamics of those countries. Note that intertemporal substitution elasticity larger than 1 (CRRA parameter is smaller than 1) is supported by some empirical studies, for example, Vissing-Jorgenson and Attanasio (2003). Under this condition ($\theta < 1$), advanced country with $\bar{\varepsilon} < \varepsilon < 1 - \alpha$, lose steady state, and as is given in Appendix, E_3 is shown to be source around $\varepsilon = 1$. These results imply that too large externality diffuses economy, so hereafter, we confine our analysis mainly to the case $0 < \varepsilon < \bar{\varepsilon}$.

4 Dynamical system and stability

Because the case without human capital accumulation (which is related to $\{E_0\}$) is reduced to the standard Ramsey model, saddle stability is easy to prove. Thus, we concentrate on the cases with positive long-run growth (related to E_1, E_2).

4.1 Dynamical equations

Let us define the following variables:

$$w(t) := (1 - \alpha)Ax(t)^{1-\alpha}u(t)^{1-\alpha}, \quad \text{and} \quad p(t) := \frac{\mu(t)}{\lambda(t)}. \quad (17)$$

Note that p corresponds to the stock price of human capital if the agents can trade their ownership of human capital in the asset market.

From r (defined in (9)) and w in (17), we obtain the following two properties:

$$r(t)^{1-\alpha}w(t)^\alpha = \alpha^{1-\alpha}(1 - \alpha)^\alpha A, \quad \text{and} \quad \frac{r(t)}{w(t)} = \frac{\alpha}{1 - \alpha} \frac{u(t)}{x(t)}. \quad (18)$$

On the other hand, substituting r and w in (18) yields

$$w(t)h(t) = b(t)p(t)h(t), \quad \text{namely} \quad w(t) = \tilde{p}(t). \quad (19)$$

where $\tilde{p} := bp$, which represents the efficiency-adjusted stock price of human capital. Substituting (19) in (18), we have the interest rate as a function of

p as follows:

$$r(t) = r(\tilde{p}(t)) := \tilde{p}(t)^{-\frac{1-\alpha}{\alpha}}, \quad \text{for } \forall t, \quad \text{and } r'(\cdot) < 0. \quad (20)$$

where the property $r'(\cdot) < 0$ comes from the well-known Stolper–Samuelson theorem. From (18)–(20), we obtain

$$u(t) = \frac{1 - \alpha}{\alpha} \frac{r(\tilde{p}(t))}{\tilde{p}(t)} x(t). \quad (21)$$

Thus, $u(t)$ is determined by \tilde{p} and x . Next, we analyze the dynamical system using the variable set $\{\tilde{p}(t), q(t), x(t)\}$.

Defining the new variable $q := c/k$, we rewrite the system constituted by (1), (2), (11), and (21) as follows:

$$\frac{\dot{q}(t)}{q(t)} = \left(\frac{1}{\theta} - \frac{1}{\alpha} \right) r(\tilde{p}(t)) - \frac{\rho}{\theta} + q(t), \quad (22)$$

$$\frac{\dot{x}(t)}{x(t)} = \left[\frac{1}{\alpha} + v(x(t), \tilde{p}(t)) \right] r(\tilde{p}(t)) - q(t) - b(x(t)). \quad (23)$$

where $v(x, \tilde{p}) := \frac{1-\alpha}{\alpha} \frac{b(x)x}{\tilde{p}}$. Note that $u^* = v^*$ holds in the steady states with positive human capital accumulation since $r^* = b^*$ holds.

Using (9), (10) with the definition of $\tilde{p}(= b\mu/\lambda)$, and specification of $b(x)(= \bar{b}x^{-\varepsilon})$, the dynamics of \tilde{p} are given as

$$\begin{aligned} \frac{\dot{\tilde{p}}(t)}{\tilde{p}(t)} &= \frac{\dot{b}(t)}{b(t)} + \frac{\dot{\mu}(t)}{\mu(t)} - \frac{\dot{\lambda}(t)}{\lambda(t)} = r(\tilde{p}(t)) - b(x(t)) - \varepsilon \frac{\dot{x}(t)}{x(t)}, \\ &= \left\{ 1 - \varepsilon \left[\frac{1}{\alpha} + v(x(t), \tilde{p}(t)) \right] \right\} r(\tilde{p}(t)) + \varepsilon q(t) - (1 - \varepsilon)b(x(t)). \end{aligned} \quad (24)$$

The three dynamical equations, (22), (23), and (24), constitute the economic system of the model.

4.2 Stability analysis

In this section, we discuss the dynamics of the model. From Eqs. (22)–(24), we obtain the linearized dynamical equations $\{\dot{\tilde{p}}, \dot{q}, \dot{x}\}$ as follows:

$$\begin{pmatrix} \dot{\tilde{p}}(t) \\ \dot{q}(t) \\ \dot{x}(t) \end{pmatrix} = J^* \begin{pmatrix} \tilde{p}(t) - \tilde{p}^* \\ q(t) - q^* \\ x(t) - x^* \end{pmatrix},$$

where

$$J^* := \begin{pmatrix} \frac{u^*}{\alpha x^*} [-\alpha + \varepsilon (1 + \frac{\alpha u^*}{1-\alpha})] \tilde{p}^* & \varepsilon \tilde{p}^* & (1-\varepsilon) \frac{\varepsilon b(x^*)}{x^*} (1-u^*) \tilde{p}^* \\ (1-\frac{\alpha}{\theta}) \frac{u^*}{\alpha x^*} q^* & q^* & 0 \\ -\frac{u^*}{\alpha} (1 + \frac{\alpha u^*}{1-\alpha}) & -x^* & b(x^*) [(1-\varepsilon)u^* + \varepsilon] \end{pmatrix}.$$

The stability around the steady state depends on the sign of the eigenvalues derived from the above linearized system. To investigate the sign of the values, we define the following characteristic equation for the above system:

$$\Gamma(\lambda) = -\lambda^3 + Tr^* \lambda^2 + B^* \lambda + Det^*,$$

where λ , Tr^* , and Det^* , respectively, denote the eigenvalue, trace, and determinant of this system. Then, B^* is derived as follows:

$$B^* = \underbrace{-u^* \left[q^* b^* + \frac{\tilde{p}^*}{x^*} (q^* - b^* u^*) \right]}_{:=B} + \varepsilon \left[2 \frac{b^* \tilde{p}^* u^*}{\alpha x^*} \left(1 + \frac{\alpha u^*}{1-\alpha} \right) - \frac{\tilde{p}^* q^* u^*}{\alpha x^*} \left(1 - \frac{\alpha}{\theta} \right) + b^* q^* (1-u^*) \right].$$

From J^* and using $q^* = \frac{r(\tilde{p}^*)}{\alpha} [1 + (1-\alpha) \frac{b(x^*) x^*}{\tilde{p}^*}] + b(x^*)$, we have Tr^* and Det^* as follows:

$$Det^* = b(x^*) \tilde{p}^* q^* \frac{u^*}{x^*} \underbrace{\left[-\{(1-\varepsilon)u^* + \varepsilon\} + \varepsilon \left(\frac{1}{\theta} + \frac{u^*}{1-\alpha} \right) \right]}_{:=\Theta}$$

$$Tr^* = \frac{r(\tilde{p}^*)}{\alpha} \left[(1-\alpha) \left(1 + \frac{b(x^*)}{\tilde{p}^*} \right) + \varepsilon \left(1 + \frac{\alpha u^*}{1-\alpha} \right) \right] + b(x^*) \{(1-\varepsilon)u^* + \varepsilon\} > 0.$$

Therefore, the sign of Θ affects the dynamical properties through the sign of the determinant.

Lemma 2-1 E_1 is saddle stable and E_2 is the source or indeterminacy.

Proof) From the Routh–Hurwitz theorem (see, for example, Benhabib and Perli (1994) and Arnold (2000)), $Det^* < 0 (> 0)$ and $Tr^* > 0$ imply the set of eigenvalues $\{+-\}$ ($\{++\}$ or $\{--\}$), and therefore, saddle stability (source or indeterminacy). Thus, the dynamical properties in the present case depend on the sign of Θ , which yields the condition

$$u^* \left\{ \begin{array}{l} > \\ < \end{array} \right\} \underline{u} \Leftrightarrow \left\{ \begin{array}{l} \text{saddle stable} \\ \text{source or indeterminacy} \end{array} \right\}. \quad (25)$$

Applying $u_2^* < \underline{u} < u_1^*$, which is derived in the proof of Lemma 1-2 to (27), we have the result that E_1 is saddle stable and E_2 is the source or indeterminacy. (Q.E.D)

This lemma implies that u_2^* is related with $Det > 0$, that is, the indeterminacy or source. By inquiring the properties, we have the following result:

Lemma 2-2 *The system converging u_2^* shows indeterminacy if ε is sufficiently near 0.*

Proof) Following the Routh–Hurwitz theorem, the indeterminacy or source depends on the following condition:

$$-B^* + \frac{Det^*}{Tr^*} \left\{ \begin{array}{l} > \\ < \end{array} \right\} 0 \iff \left\{ \begin{array}{l} \text{indeterminacy} \\ \text{source} \end{array} \right. \quad (26)$$

Since the sign of Det^* and Tr^* is positive in this case, if $B^* < 0$ is satisfied, the system shows indeterminacy. Note that $B^*(\varepsilon)$ is the liner equation with variable ε and the constant term $\bar{B} := -u [qb + \frac{\rho}{x}(q - bu)]$; therefore, if \bar{B} is negative, $B^* < 0$ holds for a sufficiently small ε . To hold $\bar{B} < 0$, it is sufficient to hold $q^* - bu^* > 0$. (22) and $r^* = b^*$ yield $q = \frac{\rho}{\theta} - (\frac{1}{\theta} - \frac{1}{\alpha})b$ and (13) yields $bu = b(1 - \frac{1}{\theta}) + \frac{\rho}{\theta}$. Substituting these equations into $q - bu$, we have $q - bu = b(\frac{1}{\alpha-1}) > 0$; thus, we obtain the result that the system converging u_2 shows indeterminacy for a sufficiently small externality, ε .(Q.E.D)

Therefore, the case with both $0 < \varepsilon < \bar{\varepsilon}$ and the existence of E_2 , the steady states yielded by E_2 show indeterminacy for a sufficiently small externality. Contrary to several previous studies⁶, sufficiently small, even negligible, externality makes the appearance of indeterminacy possible. The countries with high educational efficiency, which are supposed to be advanced economies, might grow at high growth rates, but the formation of expectations makes it difficult for the steady states to show indeterminacy properties.

Uniting the above explorations, we have the following result:

Proposition I *Under the restriction $\bar{b} > \tilde{\rho}$, the economy with higher intertemporal substitution elasticity ($\theta \in (\underline{\theta}, 1)$) and educational externality $\varepsilon \in (0, \bar{\varepsilon})$ has multiple equilibria with a mid and high growth rate. The steady state with mid-level growth (E_1) is always saddle stable and that with a high growth rate (E_2) shows potential for indeterminacy if the economy has a sufficiently small externality.*

If countries have high educational efficiency, they are presumed to be advanced economies, and given their high intertemporal substitution elasticity,

⁶Roughly speaking, indeterminacy easily emerges under larger externalities, and therefore, studies in the literature have attempted to generate indeterminacy for smaller externalities (see, for example, Lucas (1988) and Benhabib and Perli (1994)).

they are capable of achieving positive growth but not at high levels. However, the dynamical properties related to high growth are indeterminant; thus, the realization of positive growth is posed with difficulties from expectation formation.

Proposition II *Under the restrictions $\bar{b} < \tilde{\rho}$, the economy always has a no-growth steady state (E_0), and this steady state is saddle stable. The steady state with a positive growth rate (E_2) exists for $\theta \in (0, 1)$, and, at least, for the sufficient small externality, the steady state E_2 shows indeterminacy.*

Even if countries have low educational efficiency, which are assumed to be developing or underdeveloped economies, they have the capability of positive growth if there have high intertemporal substitution elasticity and slight educational externality; however, the dynamical properties related to positive growth are indeterminant, and thus, the realization of positive growth is faced with difficulties arising from expectation formation.

5 Concluding Remark

This study extended the standard Uzawa–Lucas model by adding education efficiency that is endogenously determined to examine its dynamic properties. The main results are as follows. If an economy’s intertemporal substitution elasticity is larger than 1, and the economy has a sufficiently small but non 0 externality, the economy can have multiple steady states. However, the positive steady growth for economies with lower educational efficiency (case of developing economies) or those with higher steady growth and educational efficiency (case of advanced economies) is related to indeterminacy. Therefore, the realization of positive or higher growth is subject to expectation formation under indeterminacy. Thus, when considering an economic shock, future theoretical agenda should consider introducing and inquiring the factors that are lack in the present study, namely, monetary factors, unemployment and expectation formation structures.

Furthermore, intertemporal substitution elasticity being larger than 1 is the key factor determining multiplicity in the present study. Thus, any future agenda should include empirical studies on intertemporal substitution elasticity from the viewpoint of multiplicity.

Table 2: Equilibrium Set

θ	Case I ($b > \tilde{\rho}$)	Case II ($b < \tilde{\rho}$)
$\theta < 1$	$\{E_1, E_2\}$	$\{E_0, E_2\}$
$\theta > 1$	$\{E_1\}$	$\{E_0\}$

6 Appendix

6.1 The dynamical properties of the Case III

We check the property for the case with $\varepsilon > 1 - \alpha$ (Case III). In this case, the function Φ becomes decreasing, so multiple steady states does not emerge. The pattern of steady states is determined by only the condition derived in Lemma 1-1. The obtained result is given in Tab.2 in the column of $\varepsilon > 1 - \alpha$. E_3 denotes the equilibrium with positive human capital accumulation in the Case III.

Under this restriction, the sign of Θ yields the following condition:

$$u_3^* \begin{cases} < \\ > \end{cases} \underline{u} \Leftrightarrow \begin{cases} \text{saddle stable} \\ \text{source or indeterminacy} \end{cases} . \quad (27)$$

Since $\beta < 0$ in this case, if $\theta > 1$, then $\underline{u} < 0$, so $u_3^* > 0 > \underline{u}$ always holds. Furthermore, if $\theta < 1$, then $L(\underline{u}) < 0$ holds, so that $u_3^* > \underline{u}$ is necessary for $L(u_3^*) > 0$. Thus, we always have $u_3 > 0 > \underline{u}$, so we have the following lemma:

Lemma III *The system always has $Det^* > 0$*

However, the sign of $-B + \frac{Det}{Tr}$ is ambiguous. So, the dynamical properties in this case is difficult to be settled, and hence, we have a simplifying assumption, which gives us an analytical solution and the definite properties of the model.

Assumption for simplicity For simplicity, we assume $\varepsilon = 1$.

It should be noted that the Lemma III holds on this specification, and from Fig.3, the steady state is unique; there is endogenous growth if $x_\phi < x_\psi$ and no growth if $x_\phi > x_\psi$. Under this assumption, $v(x, \tilde{p})$ is transformed into $\tilde{v}(\tilde{p}) := (1 - \alpha)\frac{\tilde{b}}{\tilde{p}}$; namely, the variable x does not affect $v(x, \tilde{p})$, which

makes the analysis easier. Furthermore, this solution is possible not because $\varepsilon = 1$ is a singular point but because the neighborhood of ε has the same dynamical properties, as shown in the latter.

Under the assumption $\varepsilon = 1$, uniting (24) and (23) provides

$$\frac{\dot{\tilde{p}}(t)}{\tilde{p}(t)} = -\frac{1-\alpha}{\alpha} r(\tilde{p}(t)) \left(1 + \frac{\bar{b}}{\tilde{p}(t)}\right) + q(t). \quad (24')$$

Thus, the system comprises (22), (23), and (24'), and the linearized dynamical equations $\{\dot{\tilde{p}}, \dot{q}, \dot{x}\}$ are obtained as follows:

$$\begin{pmatrix} \dot{\tilde{p}}(t) \\ \dot{q}(t) \\ \dot{x}(t) \end{pmatrix} = \underbrace{\begin{pmatrix} \hat{R}\hat{b}_1\tilde{p}^* & \tilde{p}^* & 0 \\ (1 - \frac{\alpha}{\theta})q^*\hat{R} & q^* & 0 \\ -\hat{R}x^*(\hat{b}_1 + \alpha) & -x^* & \bar{b}x^{*-1} \end{pmatrix}}_{:=J_1^*} \begin{pmatrix} \tilde{p}(t) - \tilde{p}^* \\ q(t) - q^* \\ x(t) - x^* \end{pmatrix}, \quad (28)$$

where $\hat{R} := \frac{1-\alpha}{\alpha^2} \frac{r^*}{\tilde{p}^*} > 0$ and $\hat{b}_1 := 1 - \alpha + \frac{\bar{b}}{\tilde{p}^*} > 0$. Thus, we have the determinant and the trace of the present study as follows:

$$Det_1^* = \frac{q^*\bar{b}x^{*-1}}{\theta} \hat{R}\Theta_1, \quad \text{and} \quad Tr_1^* = \hat{R}\hat{b}_1\tilde{p}^* + q^* + \bar{b}x^{*-1} > 0, \quad (29)$$

where $\Theta_1 := \theta\bar{b} + (1-\theta)\alpha\tilde{p}^*$. It should be noted that the sign of Θ is positive from Lemma III. Thus, we obtain the following proposition:

Proposition III *On the neighborhood of ε , the system is always unstable.*

Proof) Under the combination of $Det_1^* > 0$ and $Tr_1^* > 0$, local indeterminacy or divergence is determined by the sign of $-B_1^* + \frac{Det_1^*}{Tr_1^*}$:

$$-B_1^* + \frac{Det_1^*}{Tr_1^*} \begin{cases} < \\ > \end{cases} 0 \iff \begin{cases} \text{unstable(divergence)} \\ \text{indeterminacy} \end{cases}, \quad (30)$$

where B_1^* is defined as follows:

$$\begin{aligned} B_1^* &:= \left| \begin{array}{cc} \hat{R}\hat{b}_1\tilde{p}^* & 0 \\ -\hat{R}x^*(\hat{b}_1 + \alpha) & \bar{b}x^{*-1} \end{array} \right| + \left| \begin{array}{cc} \hat{R}\hat{b}_1\tilde{p}^* & \tilde{p}^* \\ (1 - \frac{\alpha}{\theta})q^*\hat{R} & q^* \end{array} \right| + \left| \begin{array}{cc} q^* & 0 \\ -x^* & \bar{b}x^{*-1} \end{array} \right| \\ &= \hat{R}\frac{q^*}{\theta}\Theta + q^*\bar{b}x^{*-1} + \hat{R}\bar{b}x^{*-1}\hat{b}_1\tilde{p}^*. \end{aligned} \quad (31)$$

In Eq. (31), the only variable that can possibly take a negative value is Θ , but now, when we consider the case of $\Theta > 0$, all variables constituting B_1^* are positive. Using Eqs. (29) and (31), we can calculate

$$-B_1^* + \frac{Det_1^*}{Tr_1^*} = -\frac{\hat{R}_\theta^q \Theta (\hat{R} \hat{b}_1 \tilde{p}^* + q^*)}{\hat{R} \hat{b}_1 \tilde{p}^* + q^* + \bar{b} x^{*-1}} - (q^* \bar{b} x^{*-1} + \hat{R} \bar{b} x^{*-1} \hat{b}_1 \tilde{p}^*) < 0.$$

Thus, the system is always unstable. Furthermore, by defining $\varepsilon = 1 + \varepsilon'$, we can show that the result also holds under sufficiently small ε' . (Q.E.D.)

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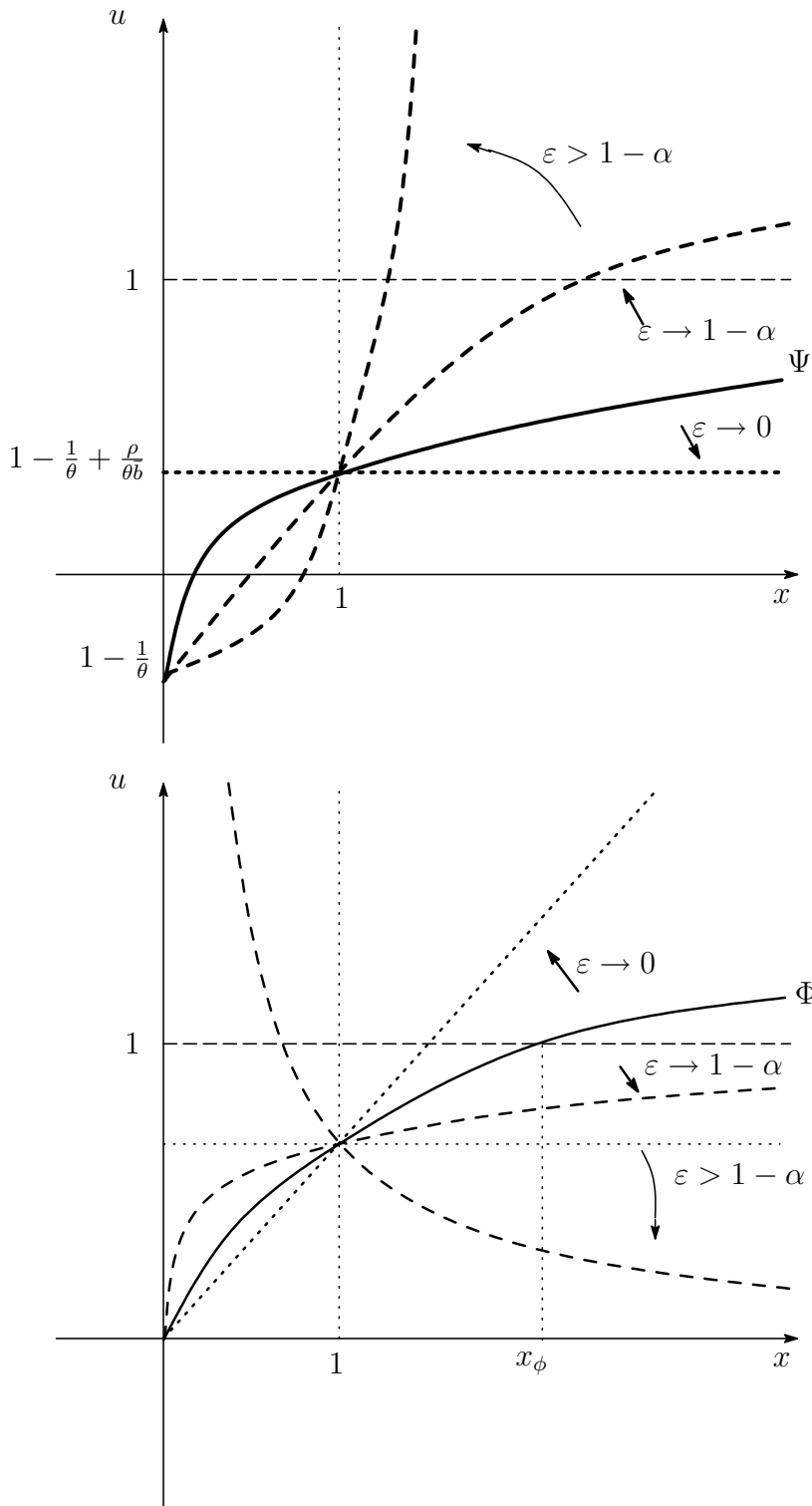


Figure 1: Form of Ψ and Φ

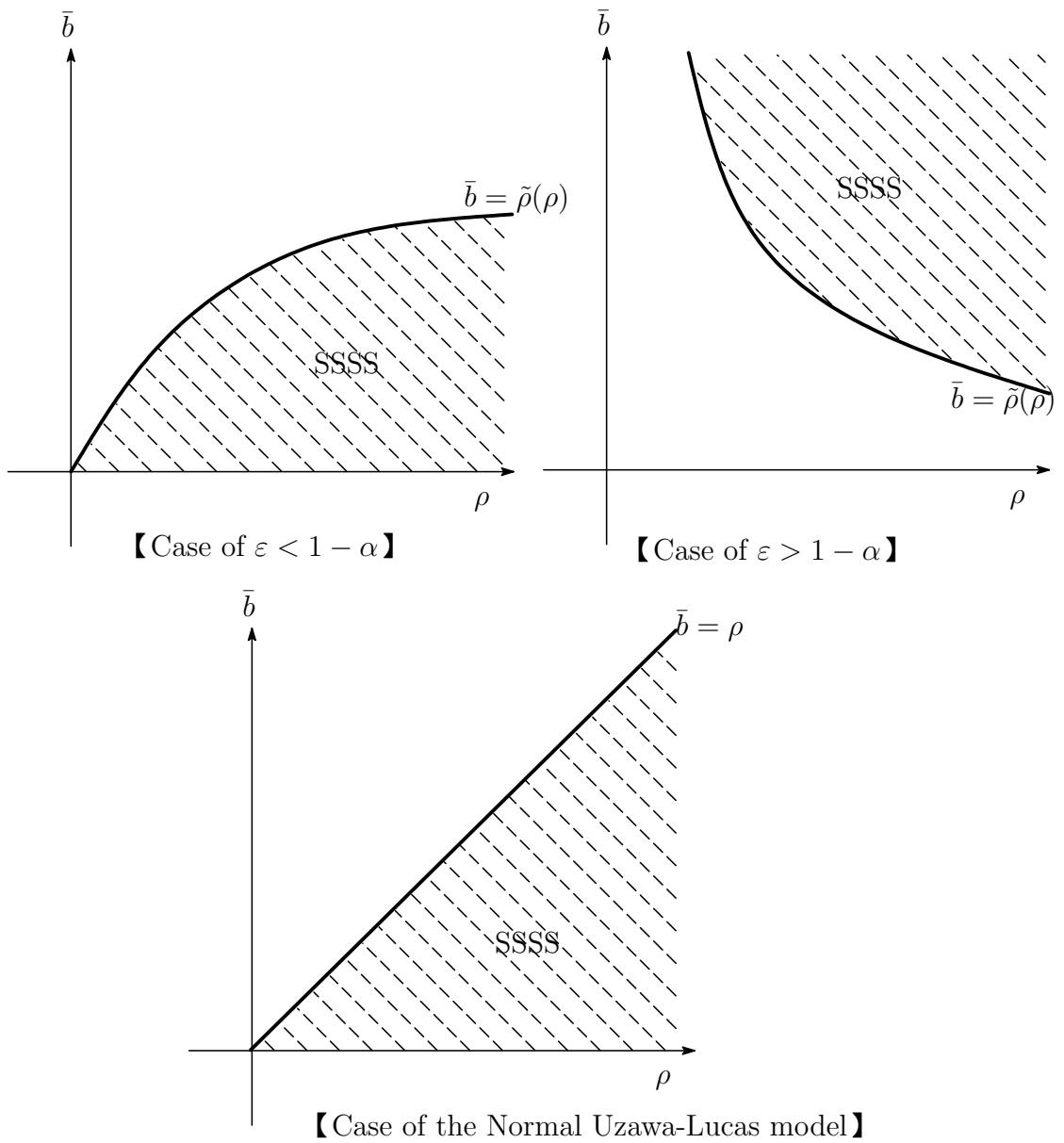


Figure 2: Determination of Steady State

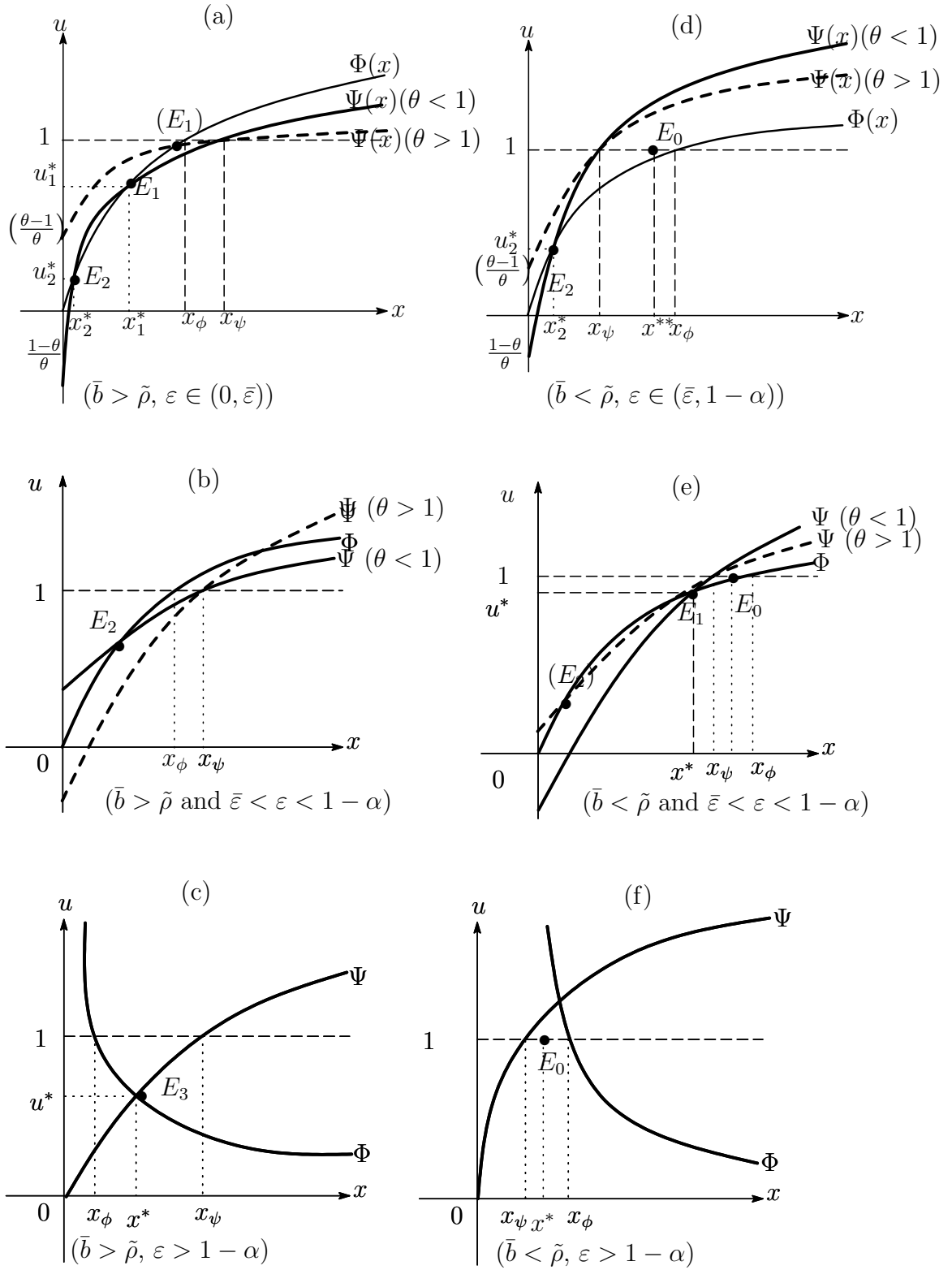
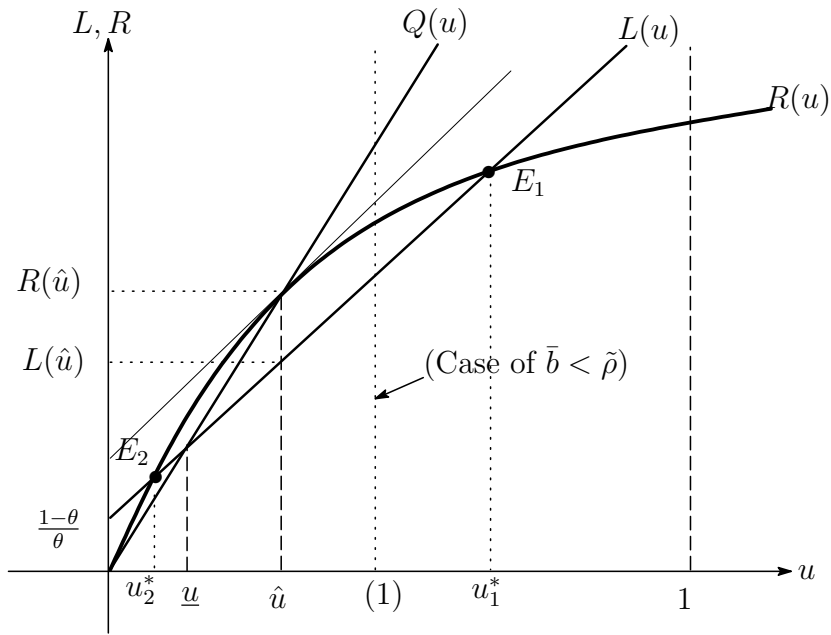
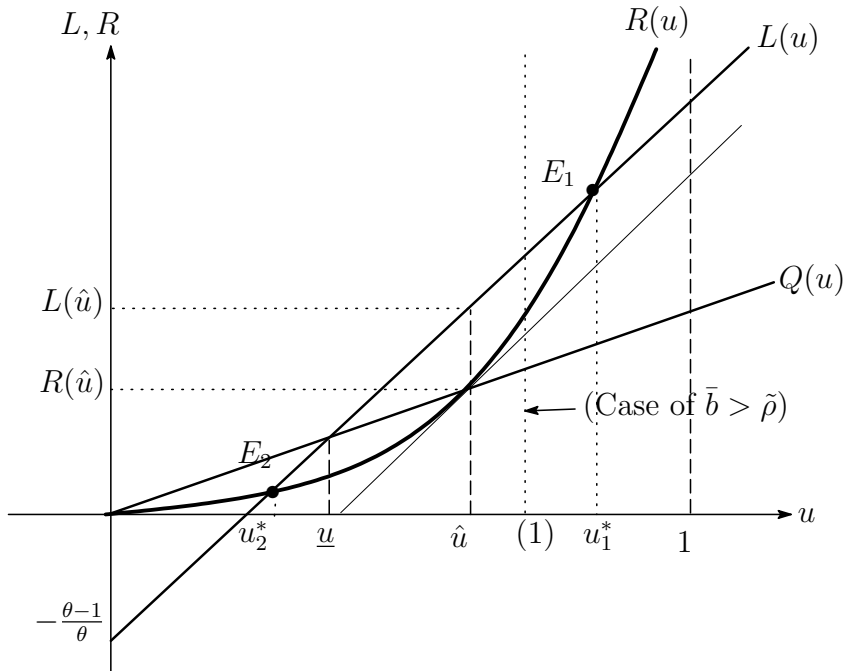


Figure 3: Equilibrium (x, u) in the Steady States

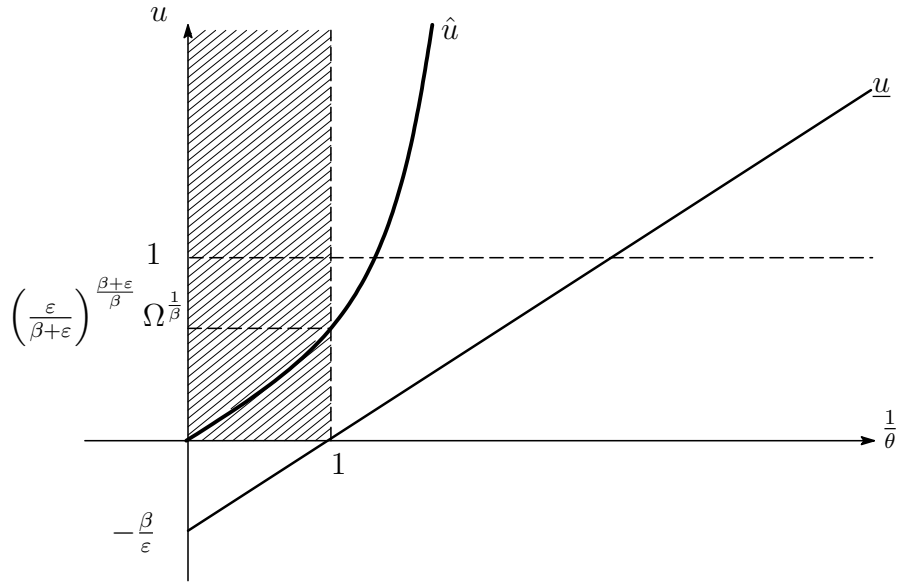


(a) Case I

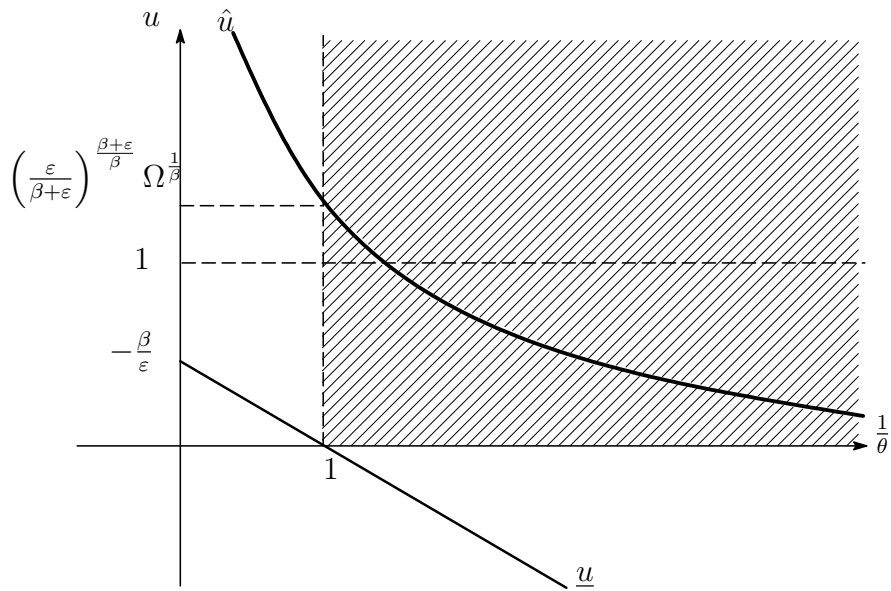


(b) Case II

Figure 4: Equilibrium u



(a) Case I ($\theta < 1$ equivalently $\frac{1}{\theta} > 1$)



(b) Case II ($\theta > 1$ equivalently $\frac{1}{\theta} < 1$)

Figure 5: \hat{u} and \underline{u}